Physics 604
Problem Set 4
Due Nov. 04, 2010

1) a) The formula in the problem follows from the expression for the potential in terms of the boundary values found in Chapter 1:

$$\Phi(r,\theta,\phi) = -\frac{1}{4\pi}\int \Phi(\vec{x}')\frac{\partial G_D}{\partial \vec{n}'}da'$$

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{|\vec{x}'|\left|\vec{x} - \frac{a^2}{|\vec{x}'|^2}\vec{x}'\right|}$$

$$\frac{\partial G_D}{\partial \vec{n}'} = \vec{n}'\cdot\frac{\partial}{\partial \vec{x}'}\left[\frac{1}{\sqrt{r^2 - 2rr'\cos\gamma + r'^2}} - \frac{a}{\sqrt{r'^2r^2 - 2a^2rr'\cos\gamma + a^4}}\right]$$

$$\left.\frac{\partial G_D}{\partial \vec{n}'}\right|_{|\vec{x}'|=a} = \frac{-a + r^2/a}{\left(r^2 - 2ra\cos\gamma + a^2\right)^{3/2}}$$

$$-\frac{1}{4\pi}\int \Phi(\vec{x}')\frac{\partial G_D}{\partial \vec{n}'}da' = \frac{a\left(a^2 - r^2\right)}{4\pi}\int \frac{\Phi(\vec{x}')}{\left(r^2 - 2ra\cos\gamma + a^2\right)^{3/2}}d\Omega'$$

We would like to manipulate this expression into something that we could apply the addition theorem to. Notice the following identity (like partial fractions expansion in calculus!)

$$\frac{1}{\sqrt{r^2 - 2ar\cos\gamma + a^2}} + 2r\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2 - 2ar\cos\gamma + a^2}} = \frac{a^2 - r^2}{\left(r^2 - 2ar\cos\gamma + a^2\right)^{3/2}}$$

Letting $\vec{x}' = a\vec{n}'$ and expanding with the addition theorem

$$\Phi(r,\theta,\phi) = \frac{a}{4\pi}\left(1 + 2r\frac{\partial}{\partial r}\right)\int \frac{\Phi(\vec{x}')}{|\vec{x} - a\vec{n}'|}d\Omega'$$

$$= \frac{a}{4\pi}\left(1 + 2r\frac{\partial}{\partial r}\right)\int \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi\Phi(\vec{x}')}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}^*(\theta',\phi')d\Omega' Y_{lm}(\theta,\phi)$$

$$= \frac{1}{4\pi}\left(1 + 2r\frac{\partial}{\partial r}\right)\int \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{\Phi(\vec{x}')}{2l+1}\frac{r^l}{a^l}Y_{lm}^*(\theta',\phi')d\Omega' Y_{lm}(\theta,\phi)$$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l}A_{lm}\frac{r^l}{a^l}Y_{lm}(\theta,\phi) \quad A_{lm} = \int \Phi(r'=a,\theta',\phi')Y_{lm}^*(\theta',\phi')d\Omega'$$

b) The other method is simply expand and match the potential at the boundary $|\vec{x}| = a$. Because the potential is not singular at the origin:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm} r^l Y_{lm}(\theta,\phi).$$

The boundary condition implies

$$\Phi(a,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_{lm} a^l Y_{lm}(\theta,\phi)$$
$$\therefore \int \Phi(a,\theta,\phi) Y_{lm}^{*}(\theta,\phi)\, d\Omega = B_{lm} a^l \equiv A_{lm}$$

by the orthonormal properties of the spherical harmonics.

2) a) The simple expression for the potential with no sphere is

$$\Phi(x,y,z) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{1}{\sqrt{x^2+y^2+(z+a)^2}} - \frac{2}{\sqrt{x^2+y^2+z^2}}\right]$$

To order a^2/r^2, the potential must be expanded through the second correction in the square root. As $a \to 0$ higher powers of a/r in the expansions become negligible

$$\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} + \frac{1}{\sqrt{x^2+y^2+(z+a)^2}}$$
$$= \frac{1}{r}\left(1 + \frac{-2az+a^2}{r^2}\right)^{-1/2} + \frac{1}{r}\left(1 + \frac{2az+a^2}{r^2}\right)^{-1/2}$$
$$= \frac{2}{r} + \frac{1}{r}\left(-\frac{1}{2}\left(\frac{-2az+a^2}{r^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{-2az+a^2}{r^2}\right)^2 + \cdots\right)$$
$$+ \frac{1}{r}\left(-\frac{1}{2}\left(\frac{2az+a^2}{r^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{2az+a^2}{r^2}\right)^2 + \cdots\right)$$
$$= \frac{1}{r}\left[-\frac{a^2}{r^2} + 3\frac{a^2}{r^2}\cos^2\theta + \cdots\right]$$
$$\Phi(r,\theta) \approx \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r^3} + 3\frac{1}{r^3}\cos^2\theta\right] = \frac{Q}{2\pi\varepsilon_0 r^3} P_2(\cos\theta).$$

b) Using the image charges for the two off-center charges, the total potential is

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2}{|\vec{x}|}+\frac{2}{b}+\frac{1}{|\vec{x}-a\hat{z}|}-\frac{b}{a|\vec{x}-b^2\hat{z}/a|}+\frac{1}{|\vec{x}+a\hat{z}|}-\frac{b}{a|\vec{x}+b^2\hat{z}/a|}\right]$$

Only $m=0$ matters, and we may expand using the addition theorem

$$\Phi(\vec{x}) = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2}{r}+\frac{2}{b}+\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{a^{l+1}r^l}{b^{2l+2}}\frac{b}{a}\right]\left[P_l(1)+P_l(-1)\right]P_l(\cos\theta)\right]$$

where $r_<$ and $r_>$ are the lesser and greater of a and r. Notice that the correct boundary condition (Φ vanishes) is in place at $r=b$. As the $l=0$ terms in the expansion cancel the first two terms above, the first non-zero potential term has $l=2$. In the limit $a\to 0$ higher powers of a/r in the expansions become negligible and

$$\Phi \to \frac{q}{4\pi\varepsilon_0}\left[\left[\frac{a^2}{r^3}-\frac{a^3r^2}{b^6}\frac{b}{a}\right]2P_l(\cos\theta)\right] = \frac{Q}{2\pi\varepsilon_0}\left[\frac{1}{r^3}-\frac{r^2}{b^5}\right]P_l(\cos\theta).$$

3) a) The first part of this problem is the hardest part. Note that $J_\nu(k\rho)$ and $J_\nu(k'\rho)$ satisfy

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k\rho)}{d\rho}\right]+\left(k^2+\frac{\nu^2}{\rho^2}\right)J_\nu(k\rho)=0$$
$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dJ_\nu(k'\rho)}{d\rho}\right]+\left(k'^2+\frac{\nu^2}{\rho^2}\right)J_\nu(k'\rho)=0$$

Multiplying the first equation by $\rho J_\nu(k'\rho)$, the second by $\rho J_\nu(k\rho)$, and subtracting shows that $\rho J_\nu(k\rho)J_\nu(k'\rho)$ can be written as a perfect differential in ρ:

$$\left(k^2-k'^2\right)\rho J_\nu(k\rho)J_\nu(k'\rho) = \frac{d}{d\rho}\left[\rho\left(J_\nu(k'\rho)\frac{dJ_\nu(k\rho)}{d\rho}-J_\nu(k\rho)\frac{dJ_\nu(k'\rho)}{d\rho}\right)\right]$$

and

$$\int_0^L \rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho = \frac{L}{\left(k^2-k'^2\right)}\left(J_\nu(k'L)\left.\frac{dJ_\nu(k\rho)}{d\rho}\right|_{\rho=L}-J_\nu(kL)\left.\frac{dJ_\nu(k'\rho)}{d\rho}\right|_{\rho=L}\right).$$

Using the recurrence relation this becomes

$$\int_0^L \rho J_\nu(k\rho)J_\nu(k'\rho)\,d\rho = \frac{1}{\left(k^2-k'^2\right)}\left(kLJ_\nu(k'L)J_{\nu+1}(kL)-k'LJ_\nu(kL)J_{\nu+1}(k'L)\right).$$

Now for given ν, k, and k', choose L large enough that the Bessel functions are well approximated by the asymptotic values

$$\int_0^L \rho J_\nu(k\rho) J_\nu(k'\rho)\, d\rho \approx \frac{2}{\pi\left(k^2-k'^2\right)\sqrt{kk'}}\begin{pmatrix} -k\cos\left(k'L-\nu\pi/2-\pi/4\right)\sin\left(kL-\nu\pi/2-\pi/4\right) \\ +k'\cos\left(kL-\nu\pi/2-\pi/4\right)\sin\left(k'L-\nu\pi/2-\pi/4\right)\end{pmatrix}$$

$$= \frac{1}{\pi\left(k^2-k'^2\right)\sqrt{kk'}}\begin{pmatrix} -k\sin\left(kL-k'L\right)+k\sin\left(kL+k'L-\nu\pi-\pi/2\right) \\ +k'\sin\left(k'L-kL\right)-k'\sin\left(k'L+kL-\nu\pi-\pi/2\right)\end{pmatrix}$$

$$= \frac{1}{\pi\left(k^2-k'^2\right)\sqrt{kk'}}\begin{pmatrix} -k\sin\left(kL-k'L\right)+k\sin\left(kL+k'L-\nu\pi-\pi/2\right) \\ +k'\sin\left(k'L-kL\right)-k'\sin\left(k'L+kL-\nu\pi-\pi/2\right)\end{pmatrix}$$

$$= \frac{1}{\pi\sqrt{kk'}}\left(\frac{\sin\left(kL-k'L\right)}{k-k'}-\left(-1\right)^\nu\frac{\cos\left(kL+k'L\right)}{k+k'}\right)$$

where the formula $\sin\left(A+B\right)+\sin\left(A-B\right)=2\sin A\cos B$ is used to simplify the terms. Now from Fourier transform theory, as discussed in class

$$\delta\left(k-k'\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(k-k')x}dx = \lim_{L\to\infty}\frac{1}{2\pi}\int_{-L}^{L} e^{i(k-k')x}dx$$

$$= \lim_{L\to\infty}\frac{1}{2\pi}\frac{e^{i(k-k')x}}{i\left(k-k'\right)}\Bigg|_{-L}^{L} = \lim_{L\to\infty}\frac{1}{\pi}\frac{\sin\left[\left(k-k'\right)L\right]}{\left(k-k'\right)}.$$

The first term goes to the proper delta-function. For the kinds of functions usually found in physics, the second term goes to zero as $L\to\infty$ because the rapidly oscillating integral averages closer and closer to zero.

b) Let the cylindrical coordinates of $\vec{x}'$ be $\left(r',\theta',z'\right)$. Then because it is the free-space Green's Function

$$\nabla^2\left[\frac{1}{\left|\vec{x}-\vec{x}'\right|}\right] = -4\pi\frac{\delta\left(r-r'\right)}{r}\delta\left(\theta-\theta'\right)\delta\left(z-z'\right)$$

Consider the two regions $z>z'$ and $z<z'$. In either region the Green Function must satisfy the Laplace Equation and taking into account it must vanish as $z\to\pm\infty$,

$$\frac{1}{\left|\vec{x}-\vec{x}'\right|} = \begin{cases} \int_0^\infty dk \sum_{m=-\infty}^{\infty} A_m(k) e^{-kz} J_m(kr) e^{im\theta} & z>z' \\ \int_0^\infty dk \sum_{m=-\infty}^{\infty} B_m(k) e^{kz} J_m(kr) e^{im\theta} & z<z' \end{cases}$$

Where the expansion “coefficient” functions $A_m(k)$ and $B_m(k)$ follow from the boundary conditions. By part a) and the continuity at $z = z'$

$$A_m(k)e^{-kz'} = B_m(k)e^{kz'}.$$

Integrating across $\delta(z-z')$ one obtains the slope change

$$\int_0^\infty dk \sum_{m=-\infty}^{\infty} (-k) A_m(k) e^{-kz} J_m(kr) e^{im\theta} - \int_0^\infty dk \sum_{m=-\infty}^{\infty} (k) B_m(k) e^{kz} J_m(kr) e^{im\theta}$$
$$= -4\pi \frac{\delta(r-r')}{r} \delta(\theta-\theta')$$
$$\therefore -A_m e^{-kz'} - B_m e^{kz'} = -2J_m(kr')e^{-im\theta'}$$
$$A_m(k) = e^{kz'} J_m(kr') e^{-im\theta'}$$
$$B_m(k) = e^{-kz'} J_m(kr') e^{-im\theta'}$$
$$\frac{1}{|\vec{x}-\vec{x}'|} = \begin{cases} \int_0^\infty dk \sum_{m=-\infty}^{\infty} e^{-k(z-z')} J_m(kr) J_m(kr') e^{im(\theta-\theta')} & z > z' \\ \int_0^\infty dk \sum_{m=-\infty}^{\infty} e^{k(z-z')} J_m(kr) J_m(kr') e^{im(\theta-\theta')} & z < z' \end{cases}$$

c) If $\vec{x}' = (r' = 0, \theta' = \theta, z' = 0)$ the expansion yields

$$\frac{1}{\sqrt{r^2+z^2}} = \int_0^\infty dk e^{-k|z|} J_0(kr),$$

because $J_m(0) = 0$ for $m \neq 0$ and $J_0(0) = 1$. Following the text example

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} = \int_0^\infty dk J_0\left(k\sqrt{r^2+r'^2-2rr'\cos\theta}\right).$$

But the denominator may also be expanded as the distance between two points with $z = z'$ and with angle θ between them

$$\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}} = \sum_{m=-\infty}^{\infty} \int_0^\infty dk e^{im\theta} J_m(kr) J_m(kr')$$

The second integrand is the superposition of orthonormal functions so the integrands must be equal in detail, yielding the second formula. I could not determine the magic limit Jackson was invoking on this problem. The result itself is a trivial consequence of the generating function for Bessel functions, but nominally we don't know about this! Here is a brute force discussion.

$$e^{ikr\cos\theta} = \sum_{n=0}^{\infty} \frac{(ikr\cos\theta)^n}{n!} = \sum_{n=0}^{\infty} i^n \left(\frac{kr}{2}\right)^n \frac{\left(e^{i\theta}+e^{-i\theta}\right)^n}{n!}$$

$$\sum_{n=0}^{\infty} i^n \left(\frac{kr}{2}\right)^n \sum_{l=0}^{n} \frac{1}{(n-l)!l!} e^{i(n-2l)\theta}$$

by the binomial expansion of the power. Consider now all terms containing $e^{im\theta}$ with $m \geq 0$. There is a term for $n = m, n = m+2, n = m+4, \cdots$ with values

$$i^m \left(\frac{kr}{2}\right)^m \frac{1}{m!0!} + i^{m+2} \left(\frac{kr}{2}\right)^{m+2} \frac{1}{(m+1)!1!} + i^{m+4} \left(\frac{kr}{2}\right)^{m+4} \frac{1}{(m+2)!2!} + \cdots$$

$$i^m \left(\frac{kr}{2}\right)^m \sum_{j=0}^{\infty} \left(\frac{kr}{2}\right)^{2j} \frac{(-1)^j}{(m+j)!j!} = i^m J_m(kr)$$

A similar discussion applies for negative integers; let $m = -|m|$. Again there is a term containing $e^{-i|m|\theta}$ for $n = |m|, n = |m|+2, n = |m|+4, \cdots$ with values

$$i^{|m|} \left(\frac{kr}{2}\right)^{|m|} \frac{1}{|m|!0!} + i^{m+2} \left(\frac{kr}{2}\right)^{|m|+2} \frac{1}{(|m|+1)!1!} + i^{m+4} \left(\frac{kr}{2}\right)^{|m|+4} \frac{1}{(|m|+2)!2!} + \cdots$$

$$i^{|m|} \left(\frac{kr}{2}\right)^{|m|} \sum_{j=0}^{\infty} \left(\frac{kr}{2}\right)^{2j} \frac{(-1)^j}{(|m|+j)!j!} = i^{|m|} J_{|m|}(kr) = i^{-m} J_{-m}(kr) = \left(\frac{-1}{i}\right)^m J_m(kr) = i^m J_m(kr)$$

Now since we have verified the formula for each power in turn

$$e^{ikr\cos\theta} = \sum_{m=-\infty}^{\infty} i^m J_m(kr) e^{im\theta}.$$

d) Notice $e^{ikr\cos\theta}$ is periodic with period 2π. Clearly from c)

$$\int_0^{2\pi} e^{ikr\cos\theta} e^{-im'\theta} d\theta = \sum_{m=-\infty}^{\infty} i^m 2\pi \delta_{mm'} J_m(kr)$$

$$\therefore J_m(kr) = \frac{1}{2\pi i^m} \int_0^{2\pi} e^{ikr\cos\theta} e^{-im\theta} d\theta$$

4) As in the exam, the 2-D polar functions which satisfy the Laplace Equation are

$$r^{\pm\nu}e^{\pm i\nu\theta}$$

The set which satisfies the Dirichlet boundary conditions at $\theta=0$ and $\theta=\beta$ are

$$r^{\pm m\pi/\beta}\sin\left(m\pi\theta/\beta\right).$$

The Green Function satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G\left(r,\theta;r',\theta'\right)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 G\left(r,\theta;r',\theta'\right)}{\partial\theta^2}=-4\pi\frac{\delta\left(r-r'\right)}{r}\delta\left(\theta-\theta'\right)$$

$$=-4\pi\frac{\delta\left(r-r'\right)}{r}\frac{2}{\beta}\sum_{m=1}^{\infty}\sin\left(m\pi\theta/\beta\right)\sin\left(m\pi\theta'/\beta\right)$$

$$G\left(r,\theta;r',\theta'\right)=\sum_{m=1}^{\infty}A_m r^{m\pi/\beta}\sin\left(m\pi\theta/\beta\right)\quad r<r'$$

$$=\sum_{m=1}^{\infty}B_m\left(r^{-m\pi/\beta}-r^{m\pi/\beta}b^{-2m\pi/\beta}\right)\sin\left(m\pi\theta/\beta\right)\quad r>r'$$

The continuity at $r=r'$ implies

$$A_m r'^{m\pi/\beta}=B_m\left(r'^{-m\pi/\beta}-r'^{m\pi/\beta}b^{-2m\pi/\beta}\right).$$

Integrating across the radial δ-function implies

$$-\frac{m\pi}{\beta}A_m r'^{m\pi/\beta}-\frac{m\pi}{\beta}B_m\left(r'^{-m\pi/\beta}+r'^{m\pi/\beta}b^{-2m\pi/\beta}\right)=-\frac{8\pi}{\beta}\sin\left(m\pi\theta'/\beta\right).$$

So

$$B_m=\frac{4}{m}\sin\left(m\pi\theta'/\beta\right)r'^{m\pi/\beta}$$

$$A_m=\frac{4}{m}\sin\left(m\pi\theta'/\beta\right)r'^{m\pi/\beta}\left(r'^{-2m\pi/\beta}-b^{-2m\pi/\beta}\right).$$

The final Green Function is

$$G\left(r,\theta;r',\theta'\right)=\sum_{m=1}^{\infty}\frac{4}{m}r_<^{m\pi/\beta}\left(r_>^{-m\pi/\beta}-r_>^{m\pi/\beta}b^{-2m\pi/\beta}\right)\sin\left(m\pi\theta/\beta\right)\sin\left(m\pi\theta'/\beta\right)$$

5) a) First recall Eqns. 1.45 and 1.46:

$$\frac{\partial G_N}{\partial \vec{n}'}(\vec{x},\vec{x}') = -\frac{4\pi}{S}$$

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}') G_N(\vec{x},\vec{x}') d^3\vec{x}' + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\partial\Phi}{\partial n'} G_N(\vec{x},\vec{x}') da'.$$

As posited, the Green Function must be of the form

$$G_N(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} g_l(r,r') \sum_{m=-l}^{l} \frac{4\pi\Phi(\vec{x}')}{2l+1} Y_{lm}^*(\theta',\phi') Y_{lm}(\theta,\phi)$$

$$g_l(r,r') = \frac{r_<^l}{r_>^{l+1}} + f_l(r,r'),$$

Where the first term of $g_l(r,r')$ yields the appropriate singularity at $r = r'$ and $f_l(r,r')$ is a solution to Laplace's Equation used to fix up the boundary conditions. Now the $\partial G_N / \partial \vec{n}'$ boundary condition is constant as a function of the angular variables, and thus affects only the $l = 0$ calculation. For $l \neq 0$

$$f_l(r,r') = A_l r^l + B_l r^{-l-1}$$

$$A_l l r^{l-1} - B_l(l+1) r^{-l-2} + l\frac{r^{l-1}}{r'^{l+1}} = 0 \qquad r = a$$

$$A_l l r^{l-1} - B_l(l+1) r^{-l-2} - (l+1)\frac{r'^l}{r^{l+2}} = 0 \qquad r = b$$

$$\therefore \begin{pmatrix} la^{l-1} & -(l+1)/a^{l+2} \\ lb^{l-1} & -(l+1)/b^{l+2} \end{pmatrix}\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \begin{pmatrix} -la^{l-1}/r'^{l+1} \\ (l+1)r'^l/b^{l+2} \end{pmatrix}$$

$$\begin{pmatrix} A_l \\ B_l \end{pmatrix} = \frac{-1}{l(l+1)\left[a^{l-1}/b^{l+2} - b^{l-1}/a^{l+2}\right]}\begin{pmatrix} -(l+1)/b^{l+2} & +(l+1)/a^{l+2} \\ -lb^{l-1} & la^{l-1} \end{pmatrix}\begin{pmatrix} -la^{l-1}/r'^{l+1} \\ (l+1)r'^l/b^{l+2} \end{pmatrix}$$

$$= \frac{-1(ab)^{l+2}}{l(l+1)\left(a^{2l+1} - b^{2l+1}\right)}\begin{pmatrix} (l+1)la^{l-1}/b^{l+2}r'^{l+1} + (l+1)^2 r'^l/a^{l+2}b^{l+2} \\ l^2 a^{l-1}b^{l-1}/r'^{l+1} + l(l+1)r'^l a^{l-1}/b^{l+2} \end{pmatrix}$$

$$\therefore f_l(r,r') = \frac{1}{b^{2l+1} - a^{2l+1}}\left[\frac{l+1}{l} r^l r'^l + \frac{(ab)^{2l+1}}{l+1}\frac{1}{(rr')^{l+1}} + a^{2l+1}\left(\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right)\right]$$

b) For $l = 0$

$$g_0(r,r') = \frac{1}{r_>} + f(r) + \frac{B_0}{r}$$

$$\left.\frac{\partial g_0}{\partial \vec{n}'}\right|_{r=a} = -\left.\frac{\partial g_0}{\partial r'}\right|_{r=a} = \frac{B_0}{a^2} = -\frac{4\pi}{4\pi\left(a^2+b^2\right)}$$

$$B_0 = -\frac{a^2}{a^2+b^2}$$

$$\left.\frac{\partial g_0}{\partial \vec{n}'}\right|_{r=b} = \left.\frac{\partial g_0}{\partial r'}\right|_{r=b} = -\frac{1+B_0}{b^2} = -\frac{4\pi}{4\pi\left(a^2+b^2\right)} = \frac{B_0}{a^2}.$$

An arbitrary $f(r)$ will solve the boundary condition. It doesn't matter in applying 1.46

$$\Phi(\vec{x}) = \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\left[\frac{1}{r_>} + f(r) - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]d^3\vec{x}'$$
$$+\frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}\left[\frac{1}{r_>} + f(r) - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]da'$$
$$= \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\left[\frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]d^3\vec{x}' + \frac{f(r)}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')d^3\vec{x}'$$
$$+\frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}\left[\frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]da' + \frac{f(r)}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}da'$$
$$= \langle\Phi\rangle_S + \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\left[\frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]d^3\vec{x}'$$
$$+\frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}\left[\frac{1}{r_>} - \frac{a^2}{a^2+b^2}\frac{1}{r'}\right]da',$$

by Gauss's Law.